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FOR SCALAR CONSERVATION LAWS**

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# REGULARIZED CHAPMAN-ENSKOG EXPANSION FOR SCALAR CONSERVATION LAWS

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## ABSTRACT

Rosenau [Phys. Rev. A, 40 (1989), pp. 7193-6] has recently proposed a regularized version of the Chapman-Enskog expansion of hydrodynamics. This regularized expansion resembles the usual Navier-Stokes viscosity terms at low wave-numbers, but unlike the latter, it has the advantage of being a bounded macroscopic approximation to the linearized collision operator.

In this paper ~~we study~~ <sup>study</sup> the behavior of Rosenau regularization of the Chapman-Enskog expansion (R-C-E) in the context of scalar conservation laws. ~~We show~~ <sup>It is shown</sup> that this R-C-E model retains the essential properties of the usual viscosity approximation, e.g., existence of travelling waves, monotonicity, upper-Lipschitz continuity etc., and at the same time, it sharpens the standard viscous shock layers. We prove that the regularized R-C-E approximation converges to the underlying inviscid entropy solution as its mean-free-path  $\epsilon \rightarrow 0$ , and we estimate the convergence rate.

(KF)

epsilon

decreases as  $\epsilon \rightarrow 0$

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## 1. Introduction

Rosenau [R] has recently proposed the scalar equation

$$\begin{aligned}
 (1.1) \quad u_t + f(u)_x &= \left[ \frac{-\varepsilon k^2}{1 + m^2 \varepsilon^2 k^2} \hat{u}(k) \right]^\vee \\
 &= \varepsilon \left[ \frac{1}{1 + m^2 \varepsilon^2 k^2} \hat{u}(k) \right]^\vee_{xx}
 \end{aligned}$$

as a model for his regularized version of the Chapman-Enskog expansion for hydrodynamics. The operator on the right side looks like the usual viscosity term  $\varepsilon u_{xx}$  at low wave-numbers  $k$ , while for higher wave numbers it is intended to model a bounded approximation of a linearized collision operator, thereby avoiding the artificial instabilities that occur when the Chapman-Enskog expansion for such an operator is truncated after a finite number of terms [R].

In this article we will compare the behavior of solutions of Rosenau regularization for the Chapman-Enskog expansion, abbreviated hereafter as the R-C-E equation, (1.1), with those of the viscous conservation law

$$(1.2) \quad u_t + f(u)_x = \varepsilon u_{xx} ,$$

towards which (1.1) tends as  $m \rightarrow 0$ , and with the conservation law with absorption

$$(1.3) \quad u_t + f(u)_x = -\frac{u}{\varepsilon m^2} .$$

Since the right side of (1.1) tends to that of (1.3) in the limit of large  $k$ , it is not surprising that the smoothness properties of solutions of the former resemble those of the latter. In particular, the R-C-E equation does not smooth out initial discontinuities, but as shown in §2, it does preserve the smoothness of initially smooth small initial data. On the other hand, the right side of (1.1) also resembles that of (1.2) in that both are second derivatives. Consequently, it is plausible that the regularized R-C-E equation (1.1), like the ordinary viscosity equation (1.2), should have travelling wave solutions connecting shock states of the underlying conservation law

$$(1.4) \quad u_t + f(u)_x = 0 .$$

In §3 we show that when  $f'' > 0$  such solutions exist iff  $m$  is sufficiently small.

At the same time, solutions of the R-C-E equation (1.1) also resemble those of the inviscid conservation law (1.4) in that both admit unique entropy solutions which share similar properties. In §4 we show that the R-C-E solution operator associated with (1.1),

like the entropy solution operator of (1.4), is  $L^1$ -contractive, monotone, and BV-bounded. Furthermore, the R-C-E solution of (1.1) tends to the inviscid entropy solution of (1.4) as the 'mean-free-path'  $\varepsilon \downarrow 0$ . Finally, if  $f'' > 0$ , the R-C-E entropy solution of (1.1) is also upper-Lipschitz continuous, in agreement with Oleinik's E-condition which characterizes the entropy solution of (1.4), and in §5 we estimate the convergence rate of the former to the latter as  $\varepsilon \downarrow 0$ .

## 2. Smoothness

It is well known that solutions of (1.2) are smooth for  $t > 0$ ; i.e. initial discontinuities are smoothed out at positive times. In contrast, by looking at piece-wise constant initial data or at the linear case  $f(u) = u$ , one sees that initial discontinuities of solutions of (1.3) are merely attenuated, not smoothed out, at positive times. Since the damping of (1.1) is less than that of (1.3), it is clear that (1.1) also does not smooth out initial discontinuities. On the other hand, if the ( $\varepsilon$ -independent) initial data for (1.3) is smooth then it will remain so provided that  $m$  is sufficiently small (see below). The next Theorem tells us that the same holds for the R-C-E equation (1.1).

**Theorem 2.1.** *The solution of the R-C-E equation (1.1) remains as smooth as its initial data,*

$$(2.1) \quad u(x, 0) = u_0(x),$$

*provided the initial data  $u_0$  are sufficiently small so that*

$$(2.2) \quad 2\{m\|u_0\|_{L^\infty}\|f''(u_0)\|_{L^\infty}\}^{1/2} + m^2\varepsilon\|f''(u_0)\|_{L^\infty}\|u_0'\|_{L^\infty} < 1.$$

**Remark.** Since we can ensure that (2.2) will be satisfied for any fixed initial data by making  $m$  sufficiently small, Theorem 2.1 can also be viewed as showing how the smoothness properties of the R-C-E equation (1.1) approach those of the viscosity equation (1.2) as  $m \rightarrow 0$ .

**Proof.** We will show formally that (2.2) implies a bound on the  $L^\infty$  norms of  $u$  and  $u_x$ . Estimates for higher derivatives then follow in standard fashion; see [M]. Furthermore, this fact ensures that the formal estimates can be justified either by smoothing the initial data or by applying a further vanishing viscosity regularization.

The first step towards obtaining the desired bounds is to note that the right side of (1.1) can be written as

$$(2.3) \quad \left[ \frac{-\varepsilon k^2}{1 + m^2 \varepsilon^2 k^2} \hat{u}(k) \right]^\vee = \frac{-1}{m^2 \varepsilon} \left\{ u - \left[ \frac{1}{1 + m^2 \varepsilon^2 k^2} \hat{u}(k) \right]^\vee \right\} \\ = \frac{-1}{m^2 \varepsilon} \{ u - Q_{m\varepsilon} * u \},$$

where  $*$  denotes convolution, and

$$(2.4) \quad Q_\varepsilon(x) \equiv \frac{1}{2\varepsilon} e^{-|x|/\varepsilon}$$

satisfies

$$(2.5) \quad \|Q_\varepsilon\|_{L^1} = 1.$$

To obtain a uniform bound on  $u$ , multiply (1.1) by  $|u^{p-2}|u$  and integrate over  $x$ ; since  $|u^{p-2}|uf(u)_x$  is an exact derivative its integral vanishes, while the contribution of the right hand-side (2.3) is nonpositive, for by (2.5),

$$(2.6) \quad \int -\frac{|u^{p-2}|}{m^2 \varepsilon} u \{ u - Q_{m\varepsilon} * u \} dx \leq \frac{-1}{m^2 \varepsilon} \{ \|u\|_{L^p}^p - \|u\|_{L^p}^{p-1} \|Q_{m\varepsilon} * u\|_{L^p} \} \\ \leq \frac{-1}{m^2 \varepsilon} \|u\|_{L^p}^p \{ 1 - \|Q_{m\varepsilon}\|_{L^1} \} = 0.$$

Dividing the remaining inequality by  $(p-1)\|u\|_{L^p}$  and integrating over  $t$  from 0 to  $T$ , we obtain

$$(2.7) \quad \|u(T)\|_{L^p} \leq \|u_0\|_{L^p},$$

and the boundedness of  $\|u(T)\|_{L^\infty}$  follows by letting  $p \uparrow \infty$ .

In order to estimate in similar fashion the  $L^\infty$  norm of  $u_x$ , we differentiate (1.1), obtaining

$$(2.8) \quad u_{xt} + f(u)_{xx} = \frac{-1}{m^2 \varepsilon} \{ u_x - Q'_{m\varepsilon} * u \},$$

and as before, we multiply (2.8) by  $|u_x|^{p-2}u_x$  and integrate over  $x$ . Integrating by parts where necessary in the term containing  $f$ , and noting that by (2.4)

$$(2.9) \quad \|Q'_{m\varepsilon}\|_{L^1} = \frac{1}{m\varepsilon},$$

we obtain after factoring out  $(p-1)\|u_x\|_{L^p}$ ,

$$(2.10) \quad \frac{d}{dt} \|u_x\|_{L^p} + \frac{1}{m^2 \varepsilon} \{ 1 - m^2 \varepsilon \|f''(u)\|_{L^\infty(t,x)} \|u_x\|_{L^\infty(t,x)} \} \|u_x\|_{L^p} \\ \leq \frac{1}{m^3 \varepsilon^2} \|u\|_{L^p} \leq \frac{1}{m^3 \varepsilon^2} \|u_0\|_{L^p}$$

Next we denote

$$(2.11) \quad Y(T) \equiv m^2 \varepsilon \|f''(u_0)\|_{L^\infty} \|u_x(T)\|_{L^\infty}.$$

Applying Gronwall's lemma to (2.10) and letting  $p \uparrow \infty$ , we obtain that  $Y \equiv \sup_T Y(T)$  does not exceed

$$(2.12) \quad \begin{aligned} Y(T) \leq Y &\leq e^{[-\frac{T(1-Y)}{m^2 \varepsilon}]} Y(0) + \frac{m}{1-Y} \{1 - e^{[-\frac{T(1-Y)}{m^2 \varepsilon}]} \} \|u_0\|_{L^\infty} \|f''(u_0)\|_{L^\infty} \\ &\leq Y(0) + \frac{m}{1-Y} \|u_0\|_{L^\infty} \|f''(u_0)\|_{L^\infty}, \end{aligned}$$

as long as  $Y < 1$ . Estimate (2.12) is a quadratic inequality for  $Y$  for which the roots of the corresponding equation are

$$(2.13) \quad Y = \frac{1}{2} \{1 + Y(0) \pm \sqrt{(1 - Y(0))^2 - 4m \|u_0\|_{L^\infty} \|f''(u_0)\|_{L^\infty}}\}.$$

Our assumption (2.2) tells us that the expression under the square root on the right is positive. Since  $Y(0)$  is bounded by the smaller root in (2.13), it follows from (2.12) plus the continuity of  $Y(T)$  that  $Y(T)$  remains bounded by this root. This in turn confirms that indeed  $Y < 1$ , and the uniform bound of  $\|u_x(T)\|_{L^\infty}$  follows.  $\square$

**Remark.** Arguing along the above lines for the conservation law with absorption (1.3), one arrives at the inequality, analogous to (2.12),  $Y(T) \leq e^{-T(1-Y)/m^2 \varepsilon} Y(0)$ . This shows that if  $Y(0) < 1$  then  $Y(T)$  remains  $< 1$ . Consequently, if  $Y(0) < 1$ , then  $Y(T)$  and hence  $\|u_x(T)\|_{L^\infty}$ , satisfy a maximum principle in this case.

### 3. Shock Profiles

Lax's generalized entropy conditions [L] for "legitimate" shock-wave solutions of the conservation law (1.4) can be interpreted as the requirement that these shocks can be realized as the limit of travelling wave solutions of the viscosity equation (1.2). If the flux function  $f$  is convex, these conditions reduce to the shock inequalities [L]

$$(3.1) \quad f'(u_-) > s > f'(u_+),$$

where  $s$  is the speed of the shock joining  $u_-$  on the left to  $u_+$  on the right. In this section we show the analogous result for the (convex) R-C-E equation (1.1): It admits travelling wave solutions whose limit as  $\varepsilon \downarrow 0$  are shock wave solutions of (1.4), iff (3.1) holds and  $m$  is sufficiently small.

**Theorem 3.1.** *Assume  $f'' > 0$ . Then (3.1) and the Rankine-Hugoniot shock condition*

$$(3.2) \quad H(u_+) = 0, \quad H(u) \equiv -s\{u - u_-\} + \{f(u) - f(u_-)\},$$

are necessary conditions for the existence of a travelling wave solution

$$u(z \equiv \frac{x-st}{\varepsilon}), \quad \lim_{z \rightarrow \pm\infty} u(z) = u_{\pm},$$

for (1.1).

Conversely, if (3.1), (3.2) hold, then a sufficient condition on  $m$  for the existence of such a travelling wave is

$$(3.3) \quad 4m^2 \sup_{u_+ < u < u_-} \{-f''(u)H(u)\} \leq 1,$$

and a necessary condition is

$$(3.4) \quad 4m^2 \{-f''(u_*)H(u_*)\} \leq 1.$$

Here  $u_*$  is defined by

$$(3.5) \quad f'(u_*) = s.$$

**Proof.** Define  $z = \frac{x-st}{\varepsilon}$  and let ' denote  $\frac{d}{dz}$ . Using (2.4) we find that a solution of (1.1) of the form  $u = u(z)$  satisfies

$$(3.6) \quad -su' + f(u)' = \{Q_m * u\}'' ,$$

where the convolution is now taken w.r.t. the variable  $z$ . The condition  $\lim_{z \rightarrow \pm\infty} u \rightarrow u_{\pm}$  implies that also  $Q_m * u \rightarrow u_{\pm}$  as  $z \rightarrow \pm\infty$ , so there exist a sequence of values  $z_j^{\pm}$  tending to  $\pm\infty$  on which  $(Q_m * u)'$  tends to zero. Hence, integrating (3.6) from  $z_j^-$  to  $z$  and letting  $j \rightarrow \infty$  we obtain

$$(3.7) \quad H(u) \equiv -s\{u - u_-\} + \{f(u) - f(u_-)\} = \{Q_m * u\}' .$$

Now letting  $z$  tend to  $+\infty$  along the sequence  $z_j^+$  we find from (3.7) that  $H(u_+) = 0$ , i.e. (3.2) holds.

Noting that  $H'' = f'' > 0$ , we see that  $H(u_-) = 0 = H(u_+)$  implies  $H' = f' - s < 0$  at the lessor of  $u_{\pm}$ , and  $H' > 0$  at the greater of the two. Hence, if  $u_+ < u_-$  then (3.1) holds, while if this inequality is reversed then so are those of (3.1), i.e., we can replace (3.1) by the condition

$$(3.8) \quad u_- > u_+ .$$

Next, we apply to (3.7) the operator  $1 - m^2 \frac{d^2}{dz^2}$  (the inverse of the operator of convolution with  $Q_m$ ), to obtain

$$(3.9) \quad u' = \{1 - m^2 \frac{d^2}{dz^2}\} H(u) = H(u) - m^2 \{H'(u)u'' + H''(u)(u')^2\}.$$

We note that since all nonzero solutions  $g$  of  $\{1 - m^2 \frac{d^2}{dz^2}\}g = 0$  are unbounded on  $R$ , the solution of (3.9) with bounded  $u$  and  $u'$  which we construct below, also satisfies (3.7). To construct such a solution we introduce the auxiliary variable

$$(3.10) \quad v = u',$$

which enables us to rewrite (3.9) as the  $2 \times 2$  system

$$(3.11) \quad u' = v$$

$$(3.12) \quad m^2 H'(u)v' = H(u) - v - m^2 H''(u)v^2.$$

The convexity of  $H(u)$  together with the Rankine-Hugoniot condition (3.2) imply that the only critical points of system (3.11-12) are  $(u_-, 0)$  and  $(u_+, 0)$ .

We remark that the linearization of (3.11-12) near the critical points  $(u_-, 0)$  and  $(u_+, 0)$  shows that they are both saddles, so that topological methods (see e.g. [S]) cannot be applied; one might even be tempted to conclude that the existence of a trajectory joining these saddle points is unlikely. What saves the day, however, is the fact that the system is singular on the line  $u = u_*$ , i.e., that the coefficient  $H'(u)$  on the left of (3.12) vanishes at  $u_*$ , which by (3.1) lies between  $u_-$  and  $u_+$ .

The key to finding a trajectory joining the two critical points is to note that solutions of (3.11-12) can cross the line  $u = u_*$  only at points  $(u_*, v_*)$  which make the right side of (3.12) vanish: Equation (3.11) implies that  $H'(u(z))$  is  $\mathcal{O}(z - z_*)$  near the value  $z_*$  for which  $u(z_*) = u_*$ , and hence (3.12) shows that  $|v| \rightarrow \infty$  as  $z \rightarrow z_*$ , unless the right side of (3.12) tends to zero. Also, since the right side of (3.12) is quadratic in  $v$ , a comparison of (3.12) with the equations  $zv' = \pm v^2$  shows that in fact  $|v|$  reaches infinity before  $u$  reaches  $u_*$ .

In order to obtain a trajectory joining  $u_-$  at  $z = -\infty$  to  $u_+$  at  $z = +\infty$  it is therefore necessary and sufficient to find trajectories joining  $(u_-, 0)$  (respectively,  $(u_+, 0)$ ) at  $z = -\infty$  (respectively,  $z = +\infty$ ) to  $(u_*, v_*)$  (with the same value of  $v_*$  for both cases) at some finite values of  $z$ ; we can always arrange for the two values of  $z$  to coincide because the system is autonomous. Since trajectories through  $(u_*, v_*)$  are not unique, the existence of our desired trajectories, which when put together join  $u_-$  to  $u_+$ , no longer seems so unlikely.



Now the right side of (3.12) is a quadratic expression in  $v$ , whose roots are

$$(3.13) \quad v_{\pm}(u) = \frac{-1 \pm \sqrt{1 + 4m^2 H''(u)H(u)}}{2m^2 H''(u)}.$$

If the argument under the square root is negative at  $u = u_*$  then clearly no such  $v_*$  exists; this gives the necessity of (3.4) for the existence of travelling wave solutions. We now turn to discuss the sufficiency of condition (3.3): it says that  $v_{\pm}$  exist for all  $u$  between  $u_-$  and  $u_+$ , and we want to show that when this happens, then the trajectories mentioned above exist iff (3.8) holds. Namely, that these trajectories exist if we replace  $u_-$  and  $u_+$  by

$$(3.14) \quad u^- = \max\{u_-, u_+\} \quad \text{and} \quad u^+ = \min\{u_-, u_+\},$$

but not with these latter two interchanged.

The linearized system around the two critical points has the form

$$(3.15) \quad \begin{pmatrix} U \\ V \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \frac{1}{m^2} & -\frac{1}{m^2 H_{\pm}'} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}.$$

Since the determinant of the matrix on the right of (3.15) is negative, both critical points are saddles, as claimed. Now, it is not hard to calculate directly the asymptotic directions of the solutions that approach each critical point as  $z$  tends to  $\pm\infty$ , as these are simply the eigenvectors of the matrix in (3.15), but in any case we will have to determine from (2.11-12) the signs of  $u'$  and  $v'$  in various regions, and this information suffices to determine which region the various asymptotic directions lie in. In this way we obtain Diagram 1, the phase-plane diagram for the case when  $m$  satisfies (3.3).

Based on Diagram 1, the existence of a travelling wave solution is argued as follows. There is a trajectory that leaves the critical point  $(u^-, 0)$  and enters region I. If this trajectory remains in region I until  $u$  reaches the value  $u_*$ , then by the above analysis it reaches the point  $(u_*, v_*)$ ; in this case,  $u' \equiv v$  as well as  $u$  are monotonic on this semi-trajectory. The only way that the trajectory can leave region I before reaching the line  $u = u_*$ , is by entering region II; but  $v' > 0$  in this region, so "clearly" the trajectory still reaches  $(u_*, v_*)$ . A similar analysis backwards in the "time"  $z$  shows that there is a semi-trajectory from  $(u_*, v_*)$  to  $(u^+, 0)$ . By checking the other trajectories leaving and entering each critical point we see that no trajectory joins  $u^+$  to  $u^-$  or either point to itself.

Although the above argument is sound provided that Diagram 1 is accurate, we have yet to verify one crucial feature of that diagram. Specifically, the argument assumed that if we enter region II at the point  $P$  and travel within this region keeping  $v' > 0$  and  $u' < 0$ , then we cannot reach region III. (Clearly, no problem arises from the possibility of re-entering region I.). Thus we have to show that the situation shown in Diagram 2 is impossible.

Diagram 1  
Phase-plane for  
small  $m$

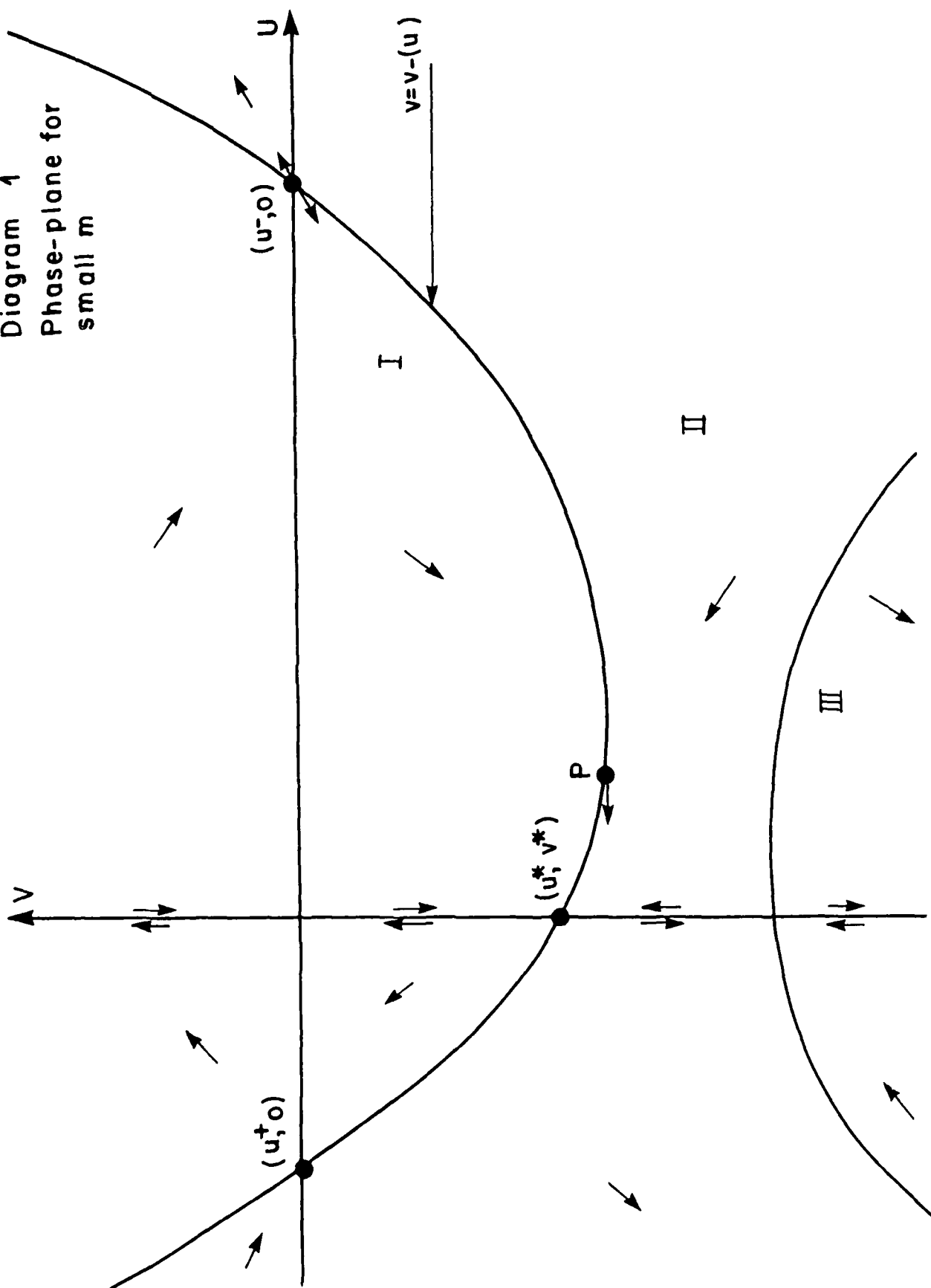
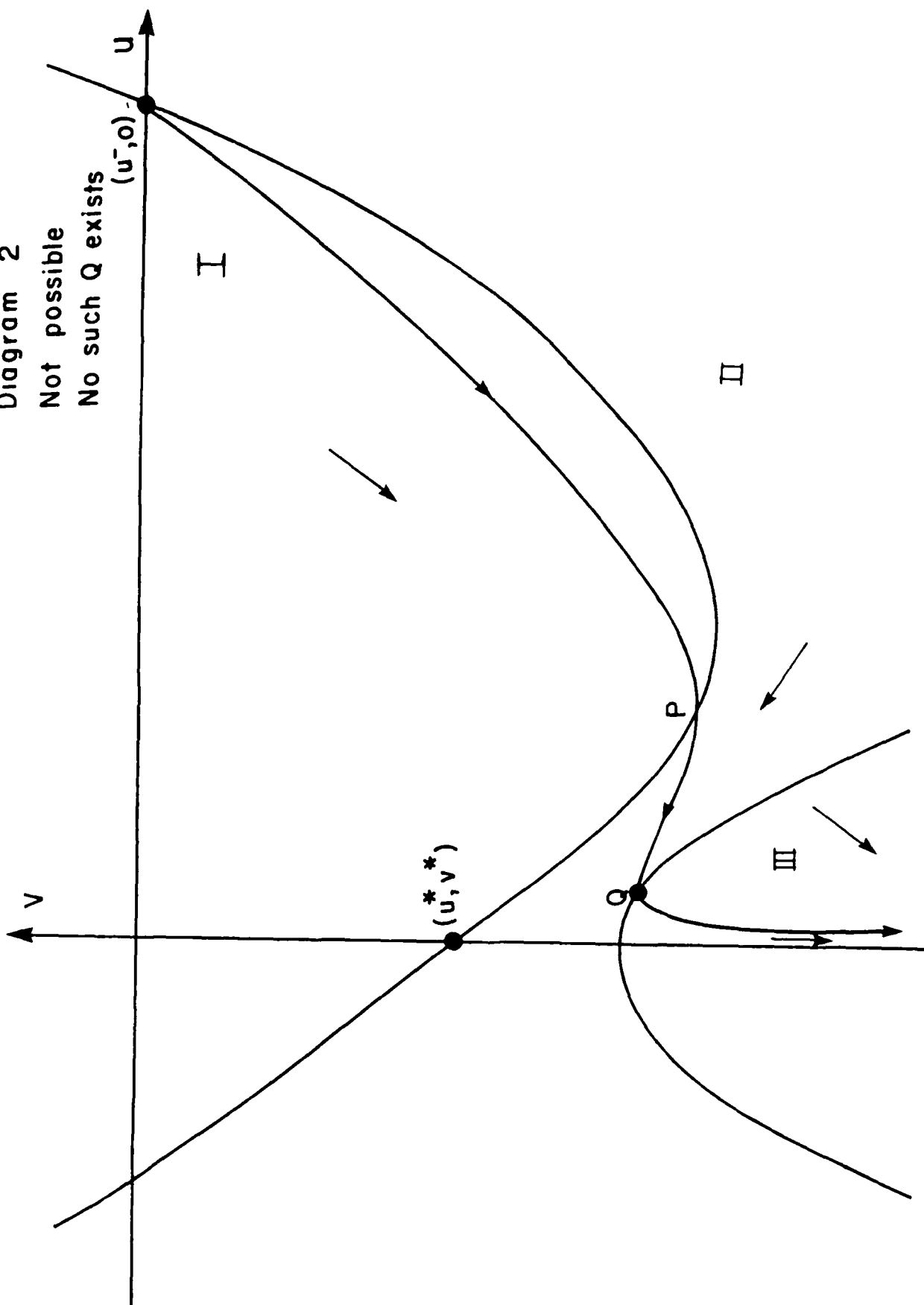


Diagram 2  
Not possible  
No such Q exists



Analytically, we must show that

$$v_-(u_1) \leq v_+(u_2) \quad \text{for } u_* < u_1 < u_2 \text{ .eqno(3.16)}$$

Defining  $H_i = -H(u_i)$  and  $A_i = -4m^2 H''(u_i)H(u_i)$ , then (3.16) boils down to

$$(3.17) \quad -2H_1\{1 + \sqrt{1 - A_1}\}/A_1 \leq -2H_2(1 - \sqrt{1 - A_2})/A_2 .$$

Now, the convexity of  $f$  (and hence of  $H$ ) together with the fact that  $H$  vanishes at  $u_{\pm}$  imply that  $H_1 > H_2 > 0$ ; these facts together with our assumption (3.3) imply that

$$(3.18) \quad 1 \geq A_i > 0 .$$

Therefore, a sufficient condition for (3.17) to hold is that

$$(3.19) \quad \{1 + \sqrt{1 - A_1}\}/A_1 \geq \{1 - \sqrt{1 - A_2}\}/A_2$$

for all  $A_i$  satisfying (3.18). A little algebraic manipulation shows that this is indeed the case. Consequently, (3.16) holds, i.e., no point such as the point  $Q$  in Diagram 2 can exist, and so the argument based on Diagram 1 is valid.

As  $m$  increases past the value that makes equality hold in (3.3), we obtain the situation shown in Diagram 3.

Namely, a gap appears in region *II*, through which our trajectory might possibly plunge into the abyss of region *III*. Hence we cannot say whether a trajectory joining  $u^-$  to  $u^+$  exists or not. Finally, when  $m$  increases past the value that makes equality hold in (3.4), then the phase-plane looks like Diagram 4, and the descent of our trajectory to  $-\infty$  becomes a certainty.  $\square$

We close this section by quantifying Rosenau's statement [R] that the travelling-wave solutions of the R-C-E equation (1.1) give narrower shock layers than those of the viscosity equation (1.4). We will adopt as our measure of shock width,  $w \equiv (u_- - u_+)/|u'_*|$  with  $u'_*$  evaluated at the point  $u_*$  at which  $H'(u) = 0$ . (It should be noted, however, that although this value of  $u'_*$  is always maximum for (1.4), it will be maximum for (1.1) only if the trajectory does not enter region *II* of Diagram 1, which in turn is guaranteed by our analysis only when the curve  $v = v_+(u)$  has its unique local minimum at  $u_*$ .) Since the relevant value of  $u'_*$  for (1.4) is given by  $u'_* = -H(u_*)$ , while the value of  $u'_*$  for (1.1) is  $v_+(u_*)$ , the estimate

$$(3.20) \quad \frac{1}{2} \leq \frac{w_{\text{Chapman-Enskog}}}{w_{\text{viscous}}} \leq 1$$

follows from the simple lemma: If a quadratic equation has real roots, then the root closer to zero lies between  $r$  and  $2r$ , where  $r$  is the root of its linear part.

Diagram3  
Phase- plane for intermediate m

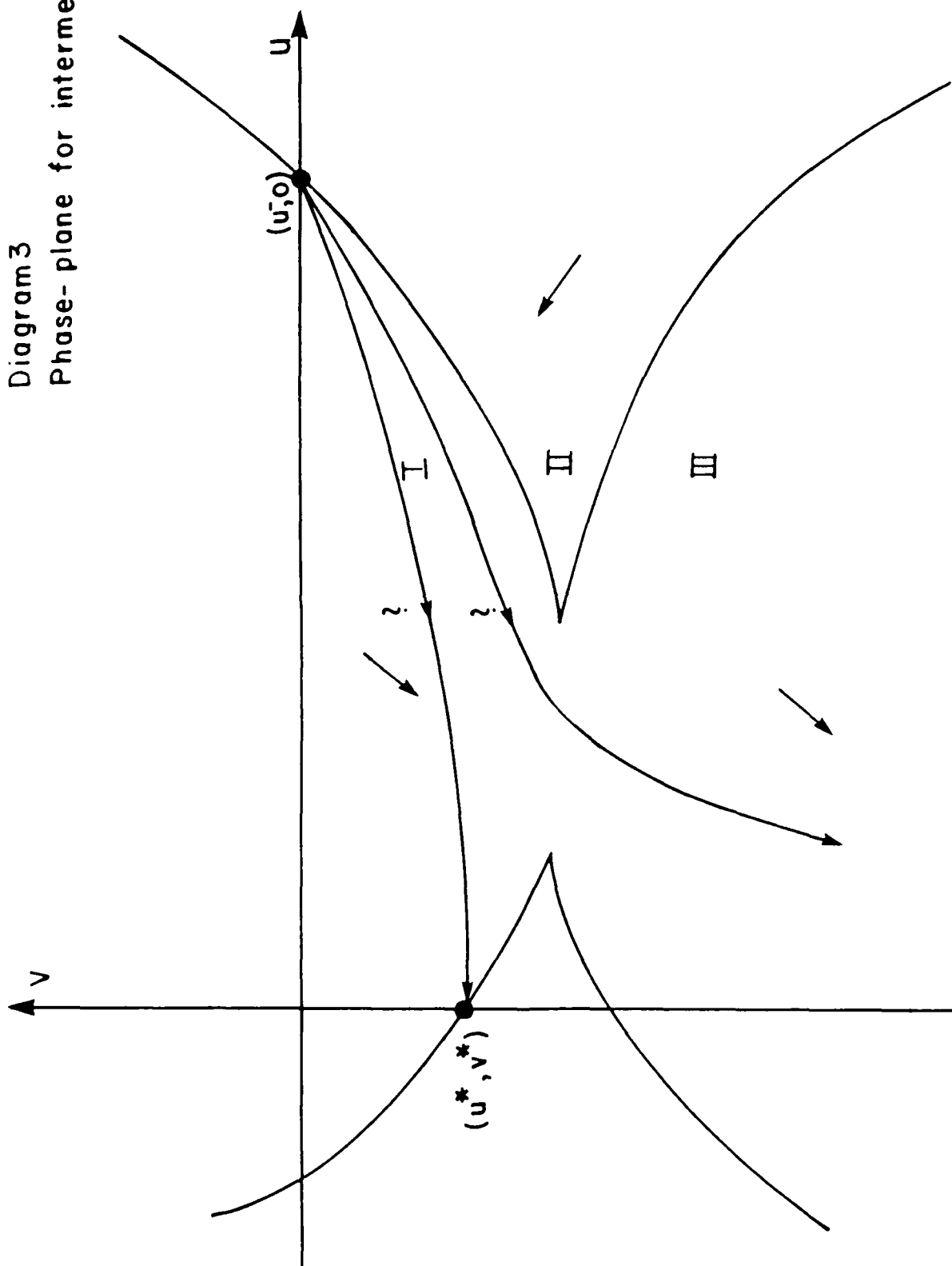
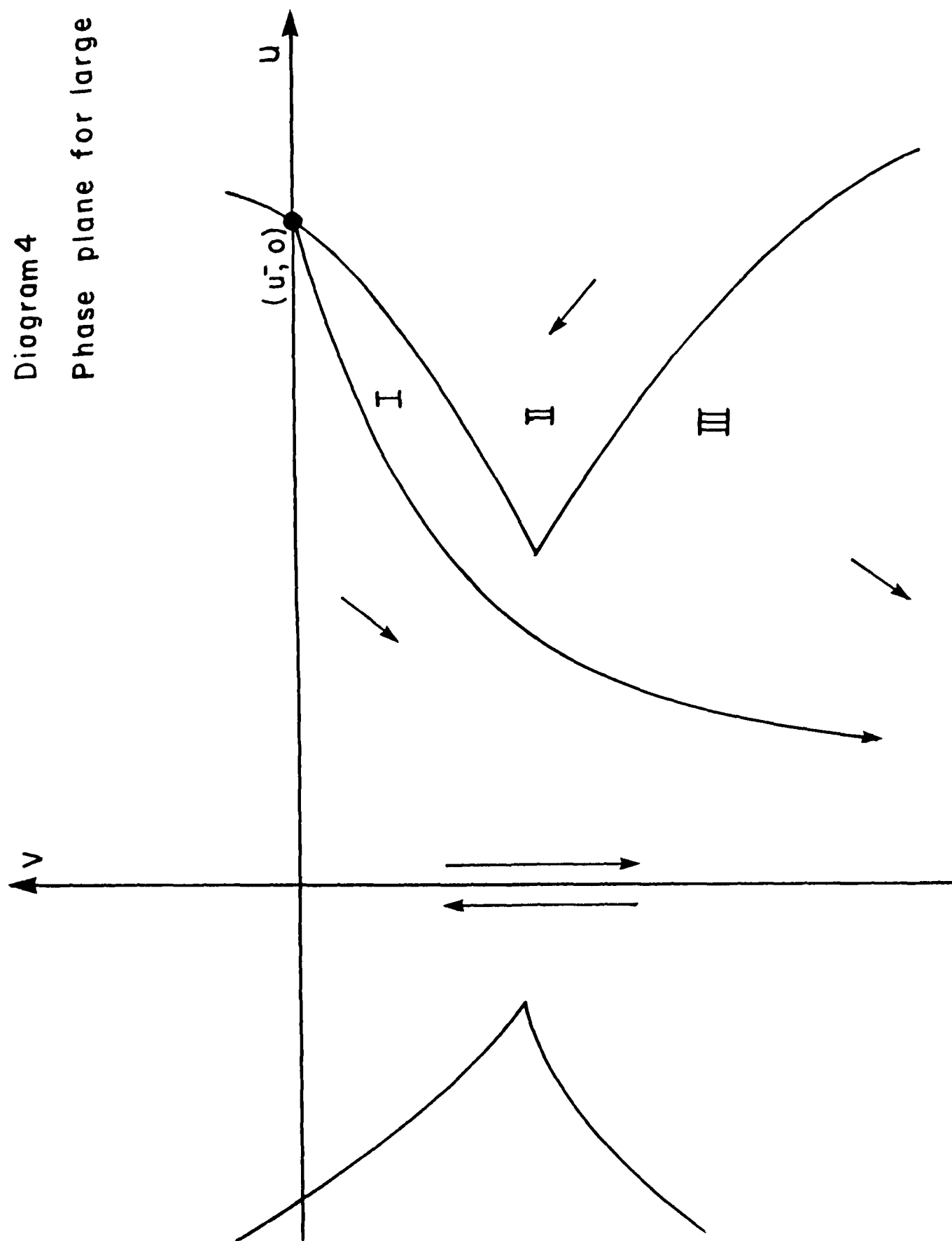


Diagram 4

Phase plane for large  $m$



#### 4. Entropy Solutions and the Zero Mean-Free-Path Limit

The parameter  $m$  does not play a role in our analysis in this section, and so will be set equal to 1 for convenience.

Since solutions of the R-C-E equation (1.1) may contain singularities weak solutions must be admitted. Since the latter need not be unique, we single out an "entropy" solution of the R-C-E equation (1.1) as the one satisfying the Krushkov-like [K] inequality

$$(4.1) \quad \begin{aligned} \partial_t |u_\epsilon - c| + \partial_x [\operatorname{sgn}(u_\epsilon - c) \{f(u_\epsilon) - f(c)\}] \\ \leq -\frac{1}{\epsilon} \{|u_\epsilon - c| - \operatorname{sgn}(u_\epsilon - c) Q_\epsilon * (u_\epsilon - c)\}, \end{aligned}$$

for all real  $c$ 's. In particular, by choosing  $c = +\sup |u_\epsilon|$  (respectively,  $c = -\sup |u_\epsilon|$ ), we obtain from (4.1) that  $u_\epsilon$  is a supersolution (respectively, a subsolution) of (1.1), and hence (1.1) is satisfied in the sense of distributions. We turn to show that (4.1) admits a unique solution  $u_\epsilon$ , and that this solution converges to the unique entropy solution of (1.4) as  $\epsilon$  goes to zero.

**Theorem 4.1.** *For any  $u_0$  in BV there exists a unique solution  $u_\epsilon$  of the R-C-E equation (4.1), and as  $\epsilon \downarrow 0$ ,  $u_\epsilon$  converges in  $L^1$  to the unique entropy solution of the inviscid conservation law (1.4).*

**Proof.**

Add the artificial viscosity term  $\delta u_{xx}$  to the right side of (1.1); the resulting equation has a unique smooth solution  $u_\epsilon^\delta$ . By a straightforward adaptation of Krushkov's [K, section 4] proof for the artificial viscosity method for (1.4), we obtain that the set  $\{u_\epsilon^\delta\}_{\delta>0}$  is bounded in BV (uniformly in  $\epsilon$  and  $\delta$ ) and precompact in  $L^1$ , and hence that a subsequence converges as  $\delta \rightarrow 0$  to a solution  $u_\epsilon$  of (4.1). Similarly, by the argument on pages 224-5 of [K] we obtain from (4.1) the consequence

$$(4.2) \quad \begin{aligned} \int_0^T \int_{-\infty}^{\infty} \{ |u_\epsilon - v_\epsilon| \Phi_t + \operatorname{sgn}(u_\epsilon - v_\epsilon) [f(u_\epsilon) - f(v_\epsilon)] \Phi_x \} dx dt \\ \geq \int_0^T \int_{-\infty}^{\infty} \frac{1}{\epsilon} \{ |u_\epsilon - v_\epsilon| - \operatorname{sgn}(u_\epsilon - v_\epsilon) Q_\epsilon * (u_\epsilon - v_\epsilon) \} \Phi dx dt \end{aligned}$$

where  $\Phi$  is an arbitrary nonnegative test function.

Next, we remark that the expression inside the curly brackets on the right of (4.2) need not be positive, but in view of (2.5), its spatial integral is. Therefore, by choosing  $\Phi(t, x) = \Phi_1(t) \Phi_2(x)$ , and letting  $\Phi_2$  tend to the function that is identically one, we obtain

$$(4.3) \quad \int_0^T \int_{-\infty}^{\infty} |u_\epsilon - v_\epsilon| \Phi_{1t} \geq 0.$$

Continuing as in [K], we let  $\Phi_1$  approach the indicator function of the interval  $[0, t]$  to conclude

$$(4.4) \quad \|u_\epsilon(t) - v_\epsilon(t)\|_{L^1} \leq \|u_\epsilon(0) - v_\epsilon(0)\|_{L^1}.$$

In particular, this shows that the solution of (4.1) is unique.

The solutions  $\{u_\epsilon\}$  of (4.1) inherit the BV bound of the  $\{u_\epsilon^\delta\}$ , and the argument of section 4 of [K] shows that this bound implies precompactness in  $L^1$ . Hence as  $\epsilon \rightarrow 0$  a subsequence converges to a weak solution  $u$  of (1.4). Because the right side of (4.2) is known to be positive only when  $\Phi$  has no dependence on  $x$ , we cannot use the entropies of (1.4) (as in [Ta]) to conclude that  $u$  is the entropy solution of (1.4). However, (4.4) implies the corresponding estimate for the weak solutions  $u$  and  $v$  obtained in the limit as  $\epsilon$  goes to zero, and by an argument of Lax [L] this suffices to show that we obtain the entropy solution: It is not hard to see that when (1.4) has a smooth solution then our scheme must converge to that solution. Hence by the corollary to theorem (3.5) of [L], any solution  $u$  of (1.4) obtained in the limit  $\epsilon \rightarrow 0$  from (4.1) has the property that all of its discontinuities satisfy the generalized Lax shock inequalities. By theorem (3.5) of [L], this implies that  $u$  is the unique entropy solution of (1.4). Finally, since any sequence of  $\epsilon s$  tending to zero has a convergent subsequence, the uniqueness of the limit shows that convergence holds without passing to a sequence.  $\square$

## 5. The Convergence Rate of the Zero Mean-Free-Path Limit

Theorem 4.1 shows that the R-C-E equation (1.1) retains several properties of the viscous conservation law (1.2). In particular, (4.4) asserts that the solution operator is an  $L^1$ -contraction, and hence by conservation plus translation invariance it is monotone [CM, Lemma 3.2], and by translation invariance it is BV-bounded:

$$(5.1) \quad \|u_\epsilon(t)\|_{BV} \leq \|u_\epsilon(0)\|_{BV}.$$

Next we show that the nonlinear R-C-E equation (1.1) also satisfies Oleinik's E-entropy condition, e.g., [Sm], [T].

**THEOREM 5.1.** *Assume  $f'' \geq \alpha > 0$ . Then the following a priori estimate holds<sup>2</sup>*

$$(5.2) \quad \|u_\epsilon(t)\|_{L^{ip+}} \leq \frac{1}{\|u_\epsilon(0)\|_{L^{ip+}}^{-1} + \alpha t}, \quad t \geq 0.$$

---

<sup>2</sup>We let  $\|\phi\|_{L^{ip}}$ ,  $\|\phi\|_{L^{ip+}}$  and  $\|\phi\|_{L^{ip}'}$  denote respectively  $\text{esssup}_{x \neq y} \left| \frac{\phi(x) - \phi(y)}{x - y} \right|$ ,  $\text{esssup}_{x \neq y} \left[ \frac{\phi(x) - \phi(y)}{x - y} \right]_+$ , and  $\text{sup}_{\psi} \frac{(\phi - \phi(0), \psi)}{\|\psi\|_{L^{ip}'}}$ .



**Remark.** The inequality (5.2) implies that the positive-variation and hence the total-variation of  $u_\epsilon(t)$  decays in time. Furthermore, this gives us another proof of the zero mean-free-path convergence to the entropy solution of (1.4) for any  $L^\infty_{loc}$ -initial data  $u_0$  (cf. Corollary 5.2).

**Proof.** We add the artificial viscosity term  $\delta u_{xx}$  to regularize (1.1), obtaining

$$(5.3) \quad \partial_t u_\epsilon^\delta + \partial_x f(u_\epsilon^\delta) = -\frac{1}{m^2 \epsilon} \{u_\epsilon^\delta - Q_{m\epsilon} * u_\epsilon^\delta\} + \delta \partial_x^2 u_\epsilon^\delta.$$

Differentiation of (5.3) yields for  $w \equiv \partial_x u_\epsilon^\delta$ ,

$$(5.4) \quad \partial_t w + f'(u_\epsilon^\delta) \partial_x w + f''(u_\epsilon^\delta) w^2 = -\frac{1}{m^2 \epsilon} \{w - Q_{m\epsilon} * w\} + \delta \partial_x^2 w.$$

Hence, since  $f'' > \alpha > 0$ , it follows that  $W(t) \equiv \max_x w(t)$  is governed by the differential inequality

$$(5.5) \quad \dot{W}(t) + \alpha W^2(t) \leq \frac{1}{m^2 \epsilon} \{W(t) - Q_{m\epsilon} * W\} \leq 0$$

and (5.2) follows by letting  $\delta \downarrow 0$ .  $\square$

Theorem 5.1 shows that solutions of the R-C-E equation (1.1) are  $Lip^+$ -stable. Moreover, (5.1) implies that the  $Lip'$ -size of their truncation is of order  $\mathcal{O}(\epsilon)$ , for

$$(5.6) \quad \|\partial_t u_\epsilon + \partial_x f(u_\epsilon)\|_{Lip'} = \epsilon \|Q_{m\epsilon} * \partial_x u_\epsilon\|_{L^1} \leq \epsilon \|Q_{m\epsilon}\|_{L^1} \|u_\epsilon(t)\|_{BV} \leq \epsilon \|u_\epsilon(0)\|_{BV}.$$

Using the result of [T] we conclude that the  $Lip'$ -convergence rate of the R-C-E solutions to the corresponding entropy solution is also of order  $\mathcal{O}(\epsilon)$ .

**Corollary 5.2.** Assume that  $f'' \geq \alpha > 0$ , and let  $u_\epsilon$  be the unique R-C-E solution of (4.1) subject to  $C^1$  initial conditions  $u_\epsilon(0) \equiv u(\cdot, 0)$ . Then  $u_\epsilon$  converges to the unique entropy solution of (1.4) and the following error estimates hold

$$\|u_\epsilon(t) - u(t)\|_{W^{-s,p}} \leq Const \cdot \epsilon^{\frac{s+1}{2p}}, \quad 1 \leq p < \infty, \quad s = 0, 1.$$

**Remark.** The choice  $(s, p) = (1, 1)$  corresponds to  $Lip'$ -convergence rate of order  $\mathcal{O}(\epsilon)$ . The choice  $(s, p) = (0, 1)$  corresponds to the usual viscous  $L^1$ -convergence rate of order  $\mathcal{O}(\epsilon^{\frac{1}{2}})$ .

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